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Dirac-Like Wave Equations for Particles of Non-Zero Rest Mass, and Their Quantization

J. S. LOMONT and H. E. MOSES

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Dirac-Like Wave Equations For Particles of Non-Zero Rest Mass, and Their Quantization

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Introduction

In the past many formulations of higher spin ($s > 1/2$) wave equations have been given. Among these several have had a more or less Dirac-like form.[1-4] These formulations all utilize higher rank undors (i.e., 4-spinors).

By utilizing an algebraic rather than an undor approach this paper will develop Dirac-like wave equations for higher spin particles in what (we believe) is the simplest possible form. The technique which will be employed is a straightforward

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1. F. J. Belinfante, Physica 6, 870 (1939).
2. W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).
3. H. A. Kramers, F. J. Belinfante, and J. K. Lubanski, Physica 8, 597 (1941).
4. H. Feshbach and W. Nichols, Annals of Physics 4, 448 (1958).

extension of the technique employed in deriving Dirac-like wave equations for massless particles.[5]

The wave equation for a particle of spin s has the form[6]

$$(\gamma_\mu \cdot \nabla^\mu + m)\psi = 0 \quad , \tag{1.1}$$

where m is the mass of the particle, the γ's are 8s x 8s square matrices satisfying the usual Dirac relations

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \quad , \tag{1.2}$$

and ψ is an 8s-component wave function in which 2s-1 components are put equal to zero. The spin 1/2 equation is the usual Dirac equation. As will be seen, this is a rather degenerate case.

The wave equations described above are covariant under the proper orthochronous Lorentz group and also under time-reversal transformations. They are not, however (except for s = 1/2), covariant under space inversion transformations. Equations invariant under space inversion can be obtained from those of type (1.1) by simply doubling the number of components of ψ. These equations are 16s-dimensional (except for s = 1/2) and again have the form (1.1) where the γ_μ matrices again satisfy (1.2).

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5. J. S. Lomont, Phys. Rev. 111, 1710 (1958).

6. Greek indices run from 0 to 3, Latin indices from 1 to 3
   $\hbar = c = 1$, $\quad -g_{00} = g_{11} = g_{22} = g_{33} = 1$,
   $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $\nabla_\mu = \partial/\partial x^\mu$,
   t and $\dagger$ mean transpose and hermitian conjugate respectively.

Finally, it should be mentioned that this approach also yields multiple spin equations of Dirac-like form.

2. General Theory

We shall begin by assuming that our wave equation has the form

$$(\gamma_\mu \nabla^\mu + m)\psi = 0 \tag{2.1}$$

where

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \quad . \tag{2.2}$$

Furthermore, we shall assume that under the infinitesimal Lorentz transformation

$$\begin{aligned} \delta x^\lambda &= \omega^{\lambda\mu} x_\mu \\ \omega^{\lambda\mu} &= -\,\omega^{\mu\lambda} \quad . \end{aligned} \tag{2.3}$$

The wave function ψ undergoes the transformation

$$\delta\psi = -\,\frac{1}{2}\,\omega^{\mu\nu} K_{\mu\nu} \psi \tag{2.5}$$

$$K_{\mu\nu} = -\,K_{\nu\mu} \quad , \tag{2.6}$$

where the infinitesimal transformations $K_{\mu\nu}$ satisfy the usual

Lorentz group conduction

$$[K_{\kappa\lambda},K_{\mu\nu}]_- = g_{\kappa\mu}K_{\lambda\nu} + g_{\lambda\nu}K_{\kappa\mu} - g_{\kappa\nu}K_{\lambda\mu} - g_{\lambda\mu}K_{\kappa\nu} \qquad (2.7)$$

and the usual covariance condition

$$[\gamma_\lambda,K_{\mu\nu}]_- = g_{\lambda\nu}\gamma_\mu - g_{\lambda\mu}\gamma_\nu \quad . \qquad (2.8)$$

The covariance of the wave equation (2.1) is now easily shown.

$$\begin{aligned}\delta\left\{(\gamma_\lambda\nabla^\lambda + m)\psi\right\} &= \gamma_\lambda(\delta\nabla^\lambda)\psi + (\gamma_\lambda\nabla^\lambda + m)\delta\psi \\ &= \gamma_\lambda\omega^{\lambda\mu}\nabla_\mu\psi - \tfrac{1}{2}(\gamma_\lambda\nabla^\lambda + m)\omega^{\mu\nu}K_{\mu\nu}\psi \\ &= \omega^{\lambda\mu}\gamma_\lambda\nabla_\mu\psi \\ &\quad - \tfrac{1}{2}\omega^{\mu\nu}K_{\mu\nu}(\gamma_\lambda\nabla^\lambda + m)\psi \\ &\quad - \tfrac{1}{2}\omega^{\mu\nu}\gamma_\mu\nabla_\nu\psi \\ &\quad + \tfrac{1}{2}\omega^{\mu\nu}\gamma_\nu\nabla_\mu\psi \\ &= 0 \quad . \end{aligned} \qquad (2.9)$$

Now we shall find all possible matrix forms of the γ's and K's by finding all irreducible representations of the γ's and K's. If we let

$$M_{\mu\nu} \equiv K_{\mu\nu} + \tfrac{1}{4}[\gamma_\mu,\gamma_\nu]_- \qquad (2.10)$$

we see that

$$[M_{\kappa\lambda}, M_{\mu\nu}]_- = g_{\kappa\mu}M_{\lambda\nu} + g_{\lambda\nu}M_{\kappa\mu} - g_{\kappa\nu}M_{\lambda\mu} - g_{\lambda\mu}M_{\kappa\nu} \qquad (2.11)$$

and

$$[\gamma_\lambda, M_{\mu\nu}]_- = 0 \quad . \qquad (2.12)$$

Consequently if

$$\gamma_\lambda \longrightarrow D(\gamma_\lambda)$$

$$M_{\mu\nu} \longrightarrow D(M_{\mu\nu})$$

is a finite-dimensional irreducible representation we see from (2.2), (2.11), and (2.12) that

$$D(\gamma_\lambda) = I^{(m,n)} \otimes \gamma_\lambda \qquad (m,n = 0,\tfrac{1}{2},1,\tfrac{3}{2}) \qquad (2.13)$$

$$D(M_{\mu\nu}) = D^{(m,n)}(M_{\mu\nu}) \otimes I^{(4)} \quad , \qquad (2.14)$$

where γ_λ on the right side of (2.13) is the usual 4 x 4 Dirac γ_λ, $I^{(m,n)}$ is the $(2m + 1)(2n + 1)$-dimensional unit matrix, $I^{(4)}$ is the 4-dimensional unit matrix, and $D^{(m,n)}(M_{\mu\nu})$ is a $(2m+1)(2n+1)$-dimensional irreducible representation $\triangle_{m,n}$ of the Lie algebra of the homogeneous Lorentz group. Let us call the representation

of the γ's and K's defined by (2.13) and (2.14) $\Gamma_{m,n}$. The dimension of $\Gamma_{m,n}$ is $4(2m + 1)(2n + 1)$.

Let $\mathcal{L}$ be the Lorentz Lie Algebra generated by the K's, and let $\Gamma^{(s)}_{m,n}$ be the representation of $\mathcal{L}$ "subduced" by restricting $\Gamma_{m,n}$ to $\mathcal{L}$. We shall now decompose $\Gamma^{(s)}_{m,n}$ into a direct sum of irreducible representations $\Delta_{m',n'}$ of $\mathcal{L}$. Let

$$\underset{\sim}{K} = (K_{23}, K_{31}, K_{12}) \qquad \underset{\sim}{M} = (M_{23}, M_{31}, M_{12}) \tag{2.15}$$

$$\underset{\sim}{\mathcal{K}} = (K_{01}, K_{02}, K_{03}) \qquad \underset{\sim}{\mathcal{M}} = (M_{01}, M_{02}, M_{03}) \tag{2.16}$$

$$C_1 = \underset{\sim}{K}^2 - \underset{\sim}{\mathcal{K}}^2 \qquad D_1 = \underset{\sim}{M}^2 - \underset{\sim}{\mathcal{M}}^2 \tag{2.17}$$

$$C_2 = \underset{\sim}{K} \cdot \underset{\sim}{\mathcal{K}} \qquad D_2 = \underset{\sim}{M} \cdot \underset{\sim}{\mathcal{M}} \tag{2.18}$$

It is then a straightforward matter to show that

$$(C_1 - D_1 + \tfrac{3}{2})^2 - 2(C_1 - D_1 + \tfrac{3}{2}) + D_1 \pm 2iD_2 = 0 \tag{2.19}$$

$$(D_2 - C_2)^2 \pm \tfrac{i}{2}(D_2 - C_2) + \tfrac{3}{16} - \tfrac{1}{4}D_1 \mp \tfrac{i}{2}D_2 = 0 \quad . \tag{2.20}$$

Since D_1 and D_2 commute with the M's and γ's they are represented by scalar matrices in $\Gamma_{m,n}$:

$$D_1 = -2\{m(m + 1) + n(n + 1)\}I \tag{2.21}$$

$$D_2 = i\{m(m + 1) - n(n + 1)\}I \quad , \tag{2.22}$$

where I is the $4(2m + 1)(2n + 1)$-dimensional unit matrix. Con-

sequently (2.19) and (2.20) are simply equations for the eigenvalues of C_1 and C_2. The eigenvalues of C_1 and C_2 are as follows.

$$\begin{array}{ll}
\underline{C_1} & \underline{C_2} \\
-2\left\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})+n(n+1)\right\} & i\left\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})-n(n+1)\right\} \\
-2\left\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})+n(n+1)\right\} & i\left\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})-n(n+1)\right\} \\
-2\left\{m(m+1)+(n+\tfrac{1}{2})(n+\tfrac{3}{2})\right\} & i\left\{m(m+1)-(n+\tfrac{1}{2})(n+\tfrac{3}{2})\right\} \\
-2\left\{m(m+1)+(n-\tfrac{1}{2})(n+\tfrac{1}{2})\right\} & i\left\{m(m+1)-(n-\tfrac{1}{2})(n+\tfrac{1}{2})\right\}
\end{array}$$

Since every irreducible representation of $\mathcal{L}$ is characterized by a pair of eigenvalues for C_1 and C_2 we see from the preceding table that

$$\Gamma^{(s)}_{m,n} = \Delta_{m+\frac{1}{2},n} \oplus \Delta_{m-\frac{1}{2},n} \oplus \Delta_{m,n+\frac{1}{2}} \oplus \Delta_{m,n-\frac{1}{2}} . \tag{2.23}$$

The infinitesimal operators K_{23}, K_{31}, and K_{12} generate a sub-algebra $\mathcal{R}$ of $\mathcal{L}$ which is the Lie algebra of the 3-dimensional rotation group. Let us denote the $(2j + 1)$-dimensional irreducible representation of $\mathcal{R}$ by Δ_j. Then if $\Delta^{(s)}_{m,n}$ is the representation of $\mathcal{R}$ subduced by the representation $\Delta_{m,n}$ of $\mathcal{L}$, we see from the Clebsch-Gordon expansion

$$\Delta^{(s)}_{m,n} = \sum_{j=|m-n|}^{m+n} \Delta_j \quad , \tag{2.24}$$

together with (2.23), (2.5), and (2.1), that the wave equation (2.1) will in general describe multiple spin particles.

We want to show that now the irreducible representation $\Gamma_{m,n}$ can be chosen in a form such that the subduced representations of both $\mathcal{L}$ and $\mathcal{R}$ are completely reduced and such that

$$\left.\begin{array}{ll} \underset{\sim}{K}^{\dagger} = -\underset{\sim}{K} & (2.25) \\ & \\ \underset{\sim}{\mathcal{K}}^{\dagger} = \underset{\sim}{\mathcal{K}} & (2.26) \end{array}\right\} \text{or} \quad K^{\dagger}_{\mu\nu} = -K^{\mu\nu}$$

$$\gamma^{\dagger}_{\mu} = \gamma^{\mu} \tag{2.27}$$

We shall subsequently refer to this as the C.R. (completely reduced) form of $\Gamma_{m,n}$ or as the C.R. representation.

One form of $\Gamma_{m,n}$ is given by (2.13) and (2.14) where $D^{(m,n)}(M_{\mu\nu})$ can be chosen as follows:

$$D^{(m,n)}(\underset{\sim}{M}) = -i(\underset{\sim}{\sigma}^{(m)} \otimes I^{(n)} + I^{(m)} \otimes \underset{\sim}{\sigma}^{(n)}) \tag{2.28}$$

$$D^{(m,n)}(\underset{\sim}{\mathcal{M}}) = -(\underset{\sim}{\sigma}^{(m)} \otimes I^{(n)} - I^{(m)} \otimes \underset{\sim}{\sigma}^{(n)}) , \tag{2.29}$$

where $\underset{\sim}{\sigma}^{(m)}$ is the irreducible $(2m + 1)$-dimensional spin vector matrix, and $I^{(m)}$ is the $(2m + 1)$-dimensional unit matrix. Since $\underset{\sim}{\sigma}^{(m)}$ is hermitian we see that $D^{(m,n)}(\underset{\sim}{M})$ is skew hermitian and $D^{(m,n)}(\underset{\sim}{\mathcal{M}})$ is hermitian. Since the 4 x 4 Dirac γ's can be chosen to satisfy (2.27) we see that $D(\gamma_{\lambda})$ given by (2.13) can also. Thus $\Gamma_{m,n}$ in the unreduced form (2.13), (2.14) can be chosen to satisfy (2.25) - (2.27). Let us consider now the $K_{\mu\nu}$ in reduced between [illegible]

[illegible]

$\bar{\Gamma}_{n,m}$ [illegible]

[illegible]

(10.1) [illegible]

[illegible]

[illegible]

[illegible] $\bar{\Gamma}_{n,m}$ [illegible]

[illegible] $\bar{\Gamma}_{n,m}$ [illegible]

$g^{(m)}$ [illegible]

[illegible]

[illegible]

where $g^{(m)}$ [illegible]

[illegible]

[illegible]

[illegible]

Then $\bar{\Gamma}_{n,m}$ [illegible]

[illegible]

form. The irreducible components of $\underset{\sim}{\mathcal{K}}$ are of the form (2.29) (with different indices) and therefore satisfy (2.26). The irreducible components of $\underset{\sim}{K}$ are - i times spin matrices. Since spin matrices are hermitian, $\underset{\sim}{K}$ satisfies (2.25). The question now is whether the transformed γ's satisfy (2.27). Since the $K_{\mu\nu}$'s are all either hermitian or skew hermitian in both the reduced and unreduced form, it follows that the transformation matrix U can be chosen to be unitary. Since

$$(U\gamma_{\mu}U^{-1})^{\dagger} = U\gamma_{\mu}^{\dagger}U^{-1} = U\gamma^{\mu}U^{-1} \tag{2.30}$$

It follows that (2.27) holds in the transformed representation.

3. Time Reversal

We shall now show that there is a 4(2m + 1)(2n + 1)-dimensional, unitary, time reversal matrix T such that

$$TT^{*} = (-1)^{2m+2n+1}I \tag{3.1}$$

$$T^{-1}\gamma^{\mu}T = \gamma_{\mu}^{*} \tag{3.2}$$

$$\left.\begin{array}{ll} T^{-1}\underset{\sim}{K}T = \underset{\sim}{K}^{*} & (3.3)\\ & \\ T^{-1}\underset{\sim}{\mathcal{K}}T = -\underset{\sim}{\mathcal{K}}^{*} & (3.4) \end{array}\right\} \text{ or } \quad T^{-1}K_{\mu\nu}T = K^{\mu\nu *}$$

where the C.R. form of $\Gamma_{m,n}$ is used. Equation (3.1)[§] assures

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[§] It should be made clear that (3.1) is a consequence of the other requirements on T and is not itself a requirement.

that two time reversals (where $\psi' = T\psi^*$) return a state to its original value. The numbers m, n characterize the irreducible representation $\Gamma_{m,n}$. Equation (3.2) assures the covariance of the wave equation.

To prove the existence of T we first observe that the two sets of matrices $\{\gamma_\lambda, \underset{\sim}{K}, \underset{\sim}{k}\}$ and $\{\gamma^{\lambda^*}, \underset{\sim}{K}^*, -\underset{\sim}{k}^*\}$ satisfy the same commutation relations. Since they also have the same Casimir operators they must be equivalent. Again since the K's and γ's are either hermitian or skew-hermitian the matrix T transforming one set into the other can be chosen to be unitary. If N denotes a generic element of the set $\{\gamma_\lambda, \underset{\sim}{K}, \underset{\sim}{k}\}$ then

$$T^{-1}NT = \varepsilon(N)N^* \qquad \varepsilon = \pm 1 \tag{3.5}$$

from which it follows that

$$[TT^*, N]_- = 0 \quad . \tag{3.6}$$

By Schur's lemma

$$TT^* = \lambda I \quad , \tag{3.7}$$

and since T is unitary and λ real [§]

[§] Multiplying (3.7) on the left by T^{-1} and on the right by T one has $T^*T = \lambda I$. Taking the complex conjugate of this and comparing with (3.7), one finds $\lambda^* = \lambda$.

$$\lambda = \pm 1 \quad . \tag{3.8}$$

We shall now construct the operator T and show that $\lambda = (-1)^{2m+2n+1}$.

It will be convenient to introduce in addition to the C.R. representation, a representation of the $M_{\mu\nu}$'s and γ_λ's of the form (2.13)-(2.14) and (2.28)-(2.29), which we shall denote by primes, such that the matrices γ'_λ are real, γ'_i $(i = 1,2,3)$ are symmetric and γ'_0 is anti-symmetric.

The operators M' are chosen so that M'_1 and M'_3 are imaginary and symmetric and M'_2 is real and anti-symmetric. Furthermore $\mathfrak{M}'_1$, $\mathfrak{M}'_3$ are real and symmetric and $\mathfrak{M}'_2$ is imaginary and anti-symmetric. Because of the hermitian character of the N's, a unitary operator V exists such that

$$VNV^{-1} = N' \quad . \tag{3.9}$$

From (2.14) and the Clebsch-Gordon expansion (2.24), it is seen that the eigenvalues of iM_2 are all integers if both m and n are integers or half-integers and are all half-odd-integers if one of the pair m, n is an integer and the other a half-odd-integer. Consequently, we see that the eigenvalues of the square of the operator

$$D = e^{-\pi M'_2} \tag{3.10}$$

are all one in the first case and minus one in the second case.

Hence

$$D^2 = (-1)^{2m+2n} I \quad . \tag{3.11}$$

Furthermore,

$$\left.\begin{aligned} D^{-1} &= e^{\pi M'_2} \\ D^* &= D \\ D^t &= D^{-1} \end{aligned}\right\} \tag{3.12}$$

From the commutation rules for $M'_{\mu\nu}$ and the symmetry properties of M and $\underset{\sim}{\mathcal{M}}$,

$$D^t M'_{\mu\nu} D = e^{\pi M'_2} M'_{\mu\nu} e^{-\pi M'_2} = - M'^{t}_{\mu\nu} = M'^{\mu\nu *} \quad . \tag{3.13}$$

We maintain that the operator $\overline{T}$ given by

$$\overline{T} = V^{-1} D \gamma'_1 \gamma'_2 \gamma'_3 V^{-1t} \tag{3.14}$$

satisfies (3.1) - (3.4) as can be proved by direct calculation, using the properties of D and the real γ' matrices.

We shall now show that the operator T is unique within a phase factor, i.e., if T_1 and T_2 are any two unitary operators satisfying (3.2) - (3.4), then $T_2 = e^{i\phi} T_1$. From (3.2) - (3.4), it follows that

$$[T_1^{-1} \overline{T}_2 , N^*]_- = 0 \quad . \tag{3.15}$$

Since N is an irreducible set,

$$T_2 = \lambda T_1 \quad . \tag{3.16}$$

Since T_2 and T_1 are unitary, λ must be a phase factor. Hence

$$T_2 = e^{i\phi} T_1 \tag{3.17}$$

where ϕ is arbitrary. Hence

$$T = e^{i\phi}\overline{T} \tag{3.18}$$

and T satisfies (3.1) also.

4. Space Inversion

The theory as developed so far does not in general possess a space inversion operator. There are, however, some cases in which it does. These cases are characterized by having $\Gamma_{m,n}$ equivalent to $\Gamma^*_{m,n}$, i.e., $\Gamma_{m,n} = \Gamma^*_{m,n}$. A necessary and sufficient condition for these cases is m = n.

Under such circumstances (m = n) we can find a unitary, hermitian space inversion matrix S such that

$$S^2 = I \tag{4.1}$$

$$SK_{\mu\nu}S = - K^{\dagger}_{\mu\nu} = K^{\mu\nu} \tag{4.2}$$

$$S\gamma_{\lambda}S = - \gamma^{\dagger}_{\lambda} = - \gamma^{\lambda} \tag{4.3}$$

and $\psi' = S\psi$ (where the C.R. form of $\Gamma_{m,n}$ is used). The first equation assures the return of ψ to its original value after two

inversions, and the third assures the covariance of the wave equation.

To prove the existence of such an S we first observe that the two sets of matrices $\left\{K^*_{\mu\nu}, \gamma^*_\lambda\right\}$ and $\left\{- K^\dagger_{\mu\nu}, - \gamma^\dagger_\lambda\right\}$ satisfy the same commutation relations and have the same Casimir operators. They are therefore equivalent. Since $\Gamma_{m,n} = \Gamma^*_{m,n}$, it follows that the two sets $\left\{K_{\mu\nu}, \gamma_\lambda\right\}$ and $\left\{- K^\dagger_{\mu\nu}, - \gamma^\dagger_\lambda\right\}$ are equivalent. We again see that the transformation matrix can be chosen unitary. Hence (letting N denote a generic element of $\left\{K_{\mu\nu}, \gamma_\lambda\right\}$) we see that there is a unitary matrix S' such that

$$S'NS'^{-1} = - N^\dagger \quad . \tag{4.4}$$

From this it follows that

$$[S'^2, N]_- = 0 \tag{4.5}$$

and hence that

$$S'^2 = \lambda I \quad . \tag{4.6}$$

Since S' is unitary $|\lambda| = 1$. We then define S by

$$S = \lambda^{-1/2} S' \quad . \tag{4.7}$$

We note that S is hermitian.

For later purposes it will be useful to give an explicit construction for S.

We recall that if A and B are any two square d-dimensional matrices, then there exists a d^2-dimensional permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^{-1} A \otimes B \mathcal{P} = B \otimes A \quad . \tag{4.8}$$

It is important to note that $\mathcal{P}$ depends on d and on the ordering of the rows and columns of the Kronecker product but not on A and B.

We recall further that

$$\underset{\sim}{M}' = -\, i(\underset{\sim}{\sigma}^{(m)} \otimes I^{(m)} + I^{(m)} \otimes \underset{\sim}{\sigma}^{(m)}) \otimes I^{(4)} \tag{4.9}$$

$$\underset{\sim}{\mathcal{M}}' = -\, (\underset{\sim}{\sigma}^{(m)} \otimes I^{(m)} - I^{(m)} \otimes \underset{\sim}{\sigma}^{(m)}) \otimes I^{(4)} \; . \tag{4.10}$$

We now let $\mathcal{P}^{(m)}$ be the $(2m + 1)^2$ dimensional permutation matrix which transforms $\underset{\sim}{\sigma}^{(m)} \otimes I^{(m)}$ into $I^{(m)} \otimes \underset{\sim}{\sigma}^{(m)}$ and vice versa. Letting $\mathcal{P}' = \mathcal{P}^{(m)} \otimes I^{(4)}$ we have

$$\mathcal{P}'^{-1} \underset{\sim}{M}' \mathcal{P} = \underset{\sim}{M}' \tag{4.11}$$

$$\mathcal{P}'^{-1} \underset{\sim}{\mathcal{M}}' \mathcal{P}' = -\underset{\sim}{\mathcal{M}}' \tag{4.12}$$

or

$$\mathcal{P}'^{-1} M'_{\mu\nu} \mathcal{P}' = M'^{\mu\nu} \quad . \tag{4.13}$$

The space inversion operator is then given by

$$S = \pm\, i\, V^{-1} \mathcal{P}' \gamma'_0 V \quad . \tag{4.14}$$

It should be noted that $\mathcal{P}'$ is real, $\mathcal{P}'^2 = I$, and $\mathcal{P}'$ commutes with γ'_0.

It can easily be shown that S as given by (4.14) satisfies (4.1) - (4.3). It can also be shown that if S' is another hermitian matrix satisfying (4.1) -(4.3) then S' = ± S.

In case $\Gamma_{m,n} \neq \Gamma^*_{m,n}$ we construct a new theory with twice the dimension of the original. Let

$$\hat{K}_{\mu\nu} = \begin{pmatrix} K_{\mu\nu} & 0 \\ 0 & K^*_{\mu\nu} \end{pmatrix} \tag{4.15}$$

$$\hat{\gamma}_\lambda = \begin{pmatrix} \gamma_\lambda & 0 \\ 0 & \gamma^*_\lambda \end{pmatrix} \tag{4.16}$$

and let $\hat{\psi}$ be a wave function transforming under $\hat{K}$

$$\delta\hat{\psi} = -\frac{1}{2}\omega^{\mu\nu}\hat{K}_{\mu\nu}\hat{\psi} \tag{4.17}$$

and satisfying

$$(\hat{\gamma}_\mu \nabla^\mu + m)\hat{\psi} = 0 \quad . \tag{4.18}$$

We now show that there exists a unitary hermitian space inversion matrix S such that

$$\hat{S}^2 = \hat{I} \tag{4.19}$$

$$\hat{S}\hat{K}_{\mu\nu}\hat{S} = -\hat{K}^\dagger_{\mu\nu} = \hat{K}^{\mu\nu} \tag{4.20}$$

$$\hat{S}\gamma_\lambda\hat{S} = -\hat{\gamma}^\dagger_\lambda = -\hat{\gamma}^\lambda \quad , \tag{4.21}$$

where $\hat{I}$ is the identity in the doubled space. Since the two sets $\left\{\hat{K}_{\mu\nu}, \hat{\gamma}_\lambda\right\}$ and $\left\{-\hat{K}^\dagger_{\mu\nu}, -\hat{\gamma}^\dagger_\lambda\right\}$ satisfy the same commutation relations and have equivalent Casimir operators they are equivalent. Hence there exists a unitary S' such that

$$\hat{S}'\hat{N}\hat{S}'^{-1} = -\hat{N}^\dagger \quad . \tag{4.22}$$

But $\Gamma_{m,n} \neq \Gamma^*_{m,n}$ implies $\left\{K_{\mu\nu}, \gamma_\lambda\right\}$ is not equivalent to $\left\{-K^\dagger_{\mu\nu}, -\gamma^\dagger_\lambda\right\}$ and hence

$$\hat{S}' = \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \tag{4.23}$$

From (4.22)

$$A_1 N^* = -N^\dagger A_1 \tag{4.24}$$

$$A_2 N = -N^{*\dagger} A_2 \quad . \tag{4.25}$$

Furthermore, since the operators N are either hermitian or anti-hermitian we write

$$N^{\dagger} = \rho(N)N \qquad (4.26)$$

where $\rho(N) = \pm 1$. From (4.24) - (4.26)

$$[A_2A_1, N^*]_- = 0 \quad . \qquad (4.27)$$

Hence

$$A_2A_1 = \lambda I \quad . \qquad (4.28)$$

From the fact that $\hat{S}'$ is unitary

$$\begin{aligned} A_2^{\dagger} &= A_2^{-1} \\ A_1^{\dagger} &= A_1^{-1} \end{aligned} \qquad (4.29)$$

Hence

$$A_2 = \lambda A_1^{\dagger} = \lambda A_1^{-1} \qquad (4.30)$$

and

$$\hat{S}'^2 = \lambda I \quad . \qquad (4.31)$$

From the fact that $\hat{S}'$ is unitary, it follows that

$$\lambda = e^{-2i\theta} \quad . \qquad (4.32)$$

We define the space inversion operator by

$$\hat{S} = e^{i\theta}\hat{S}' = \begin{pmatrix} 0 & e^{i\theta}A_1 \\ e^{-i\theta}A_1^\dagger & 0 \end{pmatrix} . \tag{4.33}$$

Clearly $\hat{S}$ is hermitian and unitary and satisfies (4.19) - (4.21). It will be useful to write $\hat{S}$ explicitly. We maintain that a matrix $\hat{\bar{S}}$ which satisfies the requirements for S is

$$\hat{\bar{S}} = \begin{pmatrix} 0 & iV^{-1}D\gamma_0'V^* \\ iV^{*-1}D^t\gamma_0'V & 0 \end{pmatrix} . \tag{4.34}$$

Let us consider any other hermitian matrix $\hat{S}$ which satisfies (4.19) - (4.21). From the fact that

$$[\hat{S}\hat{\bar{S}},N]_- = 0 \tag{4.35}$$

we see that $\hat{S}$ must have the form

$$\hat{S} = \begin{pmatrix} 0 & e^{i\theta}A \\ e^{-i\theta}A^\dagger & 0 \end{pmatrix} \tag{4.36}$$

where

$$A = iV^{-1}D\gamma_0'V^* \tag{4.37}$$

and θ is thus far arbitrary but will be fixed later.

5. Time Reversal and Space Inversion

It will be useful to obtain relations between the time reversal and space inversion matrices.

Let us first consider the undoubled theory in which the representation is equivalent to its complex conjugate. On using (3.18) and (3.14) for T and (4.14) for S we can calculate explicitly

$$T^{-1}ST = S^{*} \quad . \tag{5.1}$$

For the doubled theory we write

$$\begin{aligned} \hat{T} &= \begin{pmatrix} T & 0 \\ 0 & T^{*} \end{pmatrix} \\ \hat{T}^{-1} &= \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{*-1} \end{pmatrix} \quad . \end{aligned} \tag{5.2}$$

We should like to obtain

$$\hat{T}^{-1}\hat{S}\hat{T} = \hat{S}^{*} \tag{5.3}$$

in order to have invariant Lagrangians. Toward this end we set the hitherto arbitrary angle θ equal to ϕ and find that (5.3) is satisfied.

6. Bilinear Covariants

Let us now consider theories possessing space and time inversion transformations which satisfy (5.1) or (5.3) and, as

mentioned previously, disregard the fact that may have come from doubled theories.

By using the space inversion transformation matrix S we can now easily form bilinear covariants. Thus $\psi^{\dagger}S\psi$ is a regular scalar under the full homogeneous Lorentz group, and $\psi^{\dagger}S\gamma_{\mu}\psi$ is a 4-vector of the second kind [7] under the full homogeneous Lorentz group. Consequently, for the Lagrangian $\underset{\sim}{L}$ we can take

$$\underset{\sim}{L} = C'[\psi^{\dagger}S(\gamma^{\mu}\nabla_{\mu} + m)\psi + \text{complex conjugate}] , \tag{6.1}$$

where C' is a real constant. As a current we have

$$j_{\mu} \sim (i\psi^{\dagger}S\gamma_{\mu}\psi) \tag{6.2}$$

The Lagrangian (6.1) leads to the covariant commutation relations

$$[\psi(x),\overline{\psi}(y)] = - iC'^{-1}(\gamma^{\mu}\nabla_{\mu} - m)\triangle(x - y) , \tag{6.3}$$

$$[\psi(x),\psi(y)] = 0 \tag{6.4}$$

where

$$\overline{\psi} = \psi^{\dagger}S . \tag{6.5}$$

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7. See M. S. Watanabe, Phys. Rev. 84, 1008 (1951) or J. M. Jauch and F. Rohrlich, Theory of Photons and Electrons (Addison-Wesley Publishing Co., Inc., Cambridge, 1955) for a definition of the four kinds of tensors.

The Hamiltonian

$$\mathcal{H} = - C' \int d^3x \; \bar{\psi}(\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} + m)\psi \tag{6.6}$$

then satisfies the usual requirement

$$\dot{\psi} = i[\mathcal{H}, \psi]_- \quad . \tag{6.7}$$

7. Interaction with the Electromagnetic Field

Again we shall assume the existence of S and drop carats. Let $\eta_{\mu\nu} = - \eta_{\mu\nu}$ be a set of matrices of the same dimension as ψ satisfying

$$\left.\begin{aligned} \eta^{\dagger}_{0j} &\equiv - \eta_{0j} \quad (7.1) \\ \eta^{\dagger}_{ij} &= \eta_{ij} \quad (7.2) \end{aligned}\right\} \; \eta^{\dagger}_{\mu\nu} = \eta^{\mu\nu}$$

$$S\eta_{\mu\nu}S = \eta^{\dagger}_{\mu\nu} = \eta^{\mu\nu} \tag{7.3}$$

$$T^{-1}\eta_{\mu\nu}T = - \eta^{\mu\nu *} \tag{7.4}$$

$$\begin{aligned}[K_{\mu\nu}, \eta_{\kappa\lambda}]_- = {} & g_{\kappa\nu}\eta_{\lambda\mu} - g_{\lambda\mu}\eta_{\kappa\nu} \\ & - g_{\kappa\mu}\eta_{\lambda\mu} - g_{\lambda\nu}\eta_{\kappa\mu} \end{aligned} \tag{7.5}$$

$$\eta^{\mu\nu}\gamma_0 = \gamma_0\eta_{\mu\nu} \quad . \tag{7.6}$$

Equation (7.6) together with $[\gamma_0, \underset{\sim}{K}]_- = \underset{\sim}{\gamma}$ and (2.8) leads to the additional relations

$$[\gamma_i, \eta_{jk}]_- = 2\delta_{ik}\gamma_0\eta_{0j} - 2\delta_{ij}\gamma_0\eta_{0k} \qquad (7.7)$$

$$[\gamma_i, \eta_{0j}]_- = 2\gamma_0\eta_{i\gamma} \;. \qquad (7.8)$$

For the interacting field we can then take the Lagrangian

$$C'^{-1}\underset{\sim}{L} = \psi^\dagger S\left\{\gamma_\mu D^\mu + m + q_2\chi^{\mu\nu}\eta_{\mu\nu}\right\}\psi + \qquad (7.9)$$
$$+ \text{ complex conjugate}$$

where

$$D^\mu = \nabla^\mu - iq_1\phi^\mu \quad , \qquad (7.10)$$

q_1 and q_2 are real coupling constants, C' is a real constant, and ϕ^λ and $\chi^{\mu\nu}$ are the respectively potential and field strength of the electromagnetic field. The resulting current is

$$j_\lambda = j_\lambda^{(1)} + j_\lambda^{(2)} \qquad (7.11)$$

where

$$j_\lambda^{(1)} = \frac{\partial \underset{\sim}{L}}{\partial\phi^\lambda} = -2iq_1C'\psi^\dagger S\gamma_\lambda\psi \qquad (7.12)$$

$$j_\lambda^{(2)} = -\nabla^\kappa \frac{\partial \underset{\sim}{L}}{\partial(\nabla^\kappa\phi^\lambda)} = -4q_2C'\nabla^\kappa\psi^\dagger S\eta_{\kappa\lambda}\psi \quad . \qquad (7.13)$$

The equation for ψ is

$$[\gamma^{\mu}D_{\mu} + m + q_2\gamma_{\mu\nu}\chi^{\mu\nu}]\psi = 0 \quad . \tag{7.14}$$

8. Charge Conjugation

Again we shall consider theories with a space inversion matrix and drop carats. We shall show that there exists a unitary matrix C, the charge conjugation matrix, such that

$$C^{*}C = I \tag{8.1}$$

$$C^{-1}\gamma_{\lambda}C = \gamma_{\lambda}^{*} \tag{8.2}$$

$$C^{-1}K_{\mu\nu}C = K_{\mu\nu}^{*} \tag{8.3}$$

$$C^{-1}TC^{-1} = T^{-1} \tag{8.4}$$

$$C^{-1}SC = \pm S^{*} \quad , \tag{8.5}$$

where the C.R. representation is used again. If $\psi^{(C)} = C\psi^{*}$ then the above equations assure that when $q_2 = 0$, $\psi^{(C)}$ satisfies the wave equation with q_1 replaced by $- q_1$ (if q_1 is real).

To prove the existence of C we note first that the two sets $\left\{\gamma_{\lambda}, K_{\mu\nu}\right\}$ and $\left\{\gamma_{\lambda}^{*}, K_{\mu\nu}^{*}\right\}$ are equivalent, so there exists a unitary C satisfying

$$C^{-1}NC = N^{*} \tag{8.6}$$

and hence

$$[CC^{*}, N]_{-} = 0 \quad . \tag{8.7}$$

If the set $\left\{\gamma_\lambda, K_{\mu\nu}\right\}$ is irreducible (i.e., is not doubled) then

$$CC^* = \lambda I \quad , \tag{8.8}$$

where λ is real, and therefore ± 1.

We can actually construct C and find the sign of λ.

In the undoubled theory a unitary operator $\overline{C}$ satisfying (8.1) - (8.5) is given by

$$\overline{C} = V^{-1} \mathcal{P}' D V^* \quad . \tag{8.9}$$

Any other unitary operator C which satisfies (8.6) is given by

$$C = e^{i\alpha}\overline{C} \tag{8.10}$$

where α is arbitrary. This result is proved by using the fact $[C\overline{C}^{-1}, N]_- = 0$.

It is easily seen that (8.1) - (8.3), are satisfied. Also

$$C^{-1}SC = - S^* \quad . \tag{8.5a}$$

To obtain (8.4) we evaluate $C^{-1}TC^{-1}$ using the explicit form of the operators

$$C^{-1}TC^{-1} = - e^{-2i(\phi-\alpha)}T^{-1} \quad . \tag{8.11}$$

If we pick $\alpha = \phi - \frac{\pi}{2}$, then (8.4) follows.

In the doubled theory, every operator $\hat{C}$ which satisfies (8.1) - (8.4) is of the form

$$\hat{C} = e^{i\alpha}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} , \qquad (8.2)$$

where $\alpha = \frac{\pi}{2}(2m + 2n + 1)$.

On using (4.36) and (4.37) for S and the fact that $D^{-1} = (-1)^{2m+2n}D$ we have for the doubled theory

$$C^{-1}SC = (-1)^{(2m+2n+1)}S^{*} \qquad (8.5b)$$

In order for q_2-terms in equation (7.12) to behave properly under charge conjugation we must require

$$C^{*}\eta_{\mu\nu}C = -\eta^{*}_{\mu\nu} \quad . \qquad (8.13)$$

9. Single Spin Equations

In this section we shall see how to select from the wave equations (2.1), (2.13), (2.14) equations which can describe particles of a single spin. Let us put

$$s = m + \frac{1}{2} \qquad (9.1)$$

$$n = 0 \qquad (9.2)$$

so the decomposition (2.23) becomes

$$\Gamma^{(s)}_{s-\frac{1}{2},0} = \Delta_{s,0} \oplus \Delta_{s-\frac{1}{2},\frac{1}{2}} \oplus \Delta_{s-1,0} \quad . \tag{9.3}$$

We shall assume that $\mathcal{R}$ is in completely reduced form with the blocks in the order (9.3) from top to bottom. Since the subduced representation $\Delta^{(s)}_{s-\frac{1}{2},\frac{1}{2}}$ of $\mathcal{R}$ decomposes

$$\Delta^{(s)}_{s-\frac{1}{2},\frac{1}{2}} = \Delta_s \oplus \Delta_{s-1} \tag{9.4}$$

we shall assume that the subduced representation of $\mathcal{R}$ is completely reduced and that the blocks have the order $\Delta_s, \Delta_s, \Delta_{s-1}, \Delta_{s-1}$.

We shall now show that if the components of ψ transforming under $\Delta_{s-1,0}$ are put equal to zero that we have a single spin theory describing particles of spin s. To do this we shall consider the spin of energy eigenfunctions in a rest state, so $\psi = ue^{ip_0x^0}$, where

$$u = \begin{pmatrix} u_s \\ u'_s \\ u_{s-1} \\ 0 \end{pmatrix} \quad , \tag{9.5}$$

and

$$- i\gamma_0 u = \varepsilon u \qquad \varepsilon = \pm 1 \quad . \tag{9.6}$$

We shall now show that $u_{s-1} = 0$. For this we first note that $[\gamma_0, \underset{\sim}{K}]_- = 0$, so that

$$\gamma_0 = \begin{pmatrix} \lambda_{11} I^{(s)} & \lambda_{12} I^{(s)} & & \\ \lambda_{21} I^{(s)} & \lambda_{22} I^{(s)} & & \\ & & \lambda_{33} I^{(s-1)} & \lambda_{34} I^{(s-1)} \\ & & \lambda_{43} I^{(s-1)} & \lambda_{44} I^{(s-1)} \end{pmatrix} \qquad (9.7)$$

where the λ's are constants, the I's are unit matrices, and the omitted elements are zero. Thus

$$- i\gamma_0 u = - i \begin{pmatrix} \lambda_{11} u_s + \lambda_{12} u_s' \\ \lambda_{21} u_s + \lambda_{22} u_s' \\ \lambda_{33} u_{s-1} \\ \lambda_{43} u_{s-1} \end{pmatrix} = \varepsilon \begin{pmatrix} u_s \\ u_s' \\ u_{s-1} \\ 0 \end{pmatrix} . \qquad (9.8)$$

Consequently either $u_{s-1} = 0$ or $\lambda_{43} = 0$. If $\lambda_{43} = 0$ then the set $\{\gamma_0, K_{\mu\nu}\}$ is reduced. Since $[\gamma_0, \underset{\sim}{K}]_- = \underset{\sim}{\gamma}$ it follows that $\underset{\sim}{\gamma}$ is reduced, and hence that the entire set $\{\gamma_\lambda, K_{\mu\nu}\}$ is reduced. Since the set $\{\gamma_\lambda, K_{\mu\nu}\}$ is irreducible it follows that $u_{s-1} = 0$, and hence that u describes particles of spin s.

This scheme of setting some components of ψ equal to zero is obviously covariant under proper, orthochronous, homogeneous Lorentz transformations. The variance of the scheme under improper transformations is not so obvious and requires proof. From (2.17), (3.3), and (3.4) we find that

$$[T, C_1]_- = 0 \quad . \qquad (9.9)$$

Consequently T must have the same block form as the $K_{\mu\nu}$'s, and

thus if the last (2s - 1) elements of ψ are zero so are the last (2s - 1) elements of $T\psi^*$.

Since the only undoubled theory with a space inversion transformation is the Dirac theory, s = 1/2, which has no zero components, we shall consider only doubled theories s > 1/2. From (2.17), and (4.20) we find that

$$[S,C_1]_- = 0 \quad . \tag{9.10}$$

Since

$$S = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \qquad C_1 = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix} \tag{9.11}$$

we find that

$$[A,C_1]_- = 0$$

from which it follows that A has the same block structure as $K_{\mu\nu}$. If $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where the last 2s - 1 components of both ψ_1 and ψ_2 are zero, then ψ_1' and ψ_2' given by $\psi' = S\psi = \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix}$ have the same property.

Finally the simple form of charge conjugation operator (8.2) assures us that the last 2s-1 components remain zero under charge conjugation.

10. Single Spin Covariant Commutation Rules

Let us consider a complete set of commuting observables consisting of $H = - i\gamma_0(\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} + m)$, $- i\underset{\sim}{K} \cdot \underset{\sim}{\nabla}$, $P = - i\underset{\sim}{\nabla}$, and

other hermitian operators which can always be chosen so that they commute with the operator $iS\gamma_0$. A simultaneous eigenstate of all these operators will be designated by

$$\psi(\underset{\sim}{x}|\underset{\sim}{p},\varepsilon,\zeta) \tag{10.1}$$

where $\underset{\sim}{p}$ is the momentum, ε is the sign of energy and ζ collectively denotes all the other eigenvalues. We shall separate out the $\underset{\sim}{x}$-dependence in the usual way

$$\psi(\underset{\sim}{x}|\underset{\sim}{p},\varepsilon,\zeta) = \frac{e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}}{(2\pi)^{3/2}}\,u(\underset{\sim}{p},\varepsilon,\zeta) \tag{10.2}$$

where u is a column vector.

For many purposes one might be inclined to normalize u so that

$$u^\dagger(\underset{\sim}{p},\varepsilon,\zeta)u(\underset{\sim}{p},\varepsilon',\zeta') = \delta_{\varepsilon\varepsilon'}\delta_{\zeta\zeta} \;. \tag{10.3}$$

This normalization leads to

$$\int\psi^\dagger(\underset{\sim}{x}|\underset{\sim}{p},\varepsilon,\zeta)\psi(\underset{\sim}{x}|\underset{\sim}{p}',\varepsilon',\zeta')d^3x = \delta(\underset{\sim}{p} - \underset{\sim}{p}')\delta_{\varepsilon\varepsilon'}\delta_{\zeta\zeta} \;. \tag{10.4}$$

However, this normalization is not covariant. It can be shown that the following covariant normalization is possible for $\mu = \varepsilon'$.

$$i\bar{u}(\underset{\sim}{p},\varepsilon,\zeta)\gamma_\mu u(\underset{\sim}{p},\varepsilon,\zeta') = \lambda(\varepsilon,\zeta)p_\mu\delta_{\zeta\zeta'} \tag{10.5}$$

where here

$$\left.\begin{aligned} p_0 &= -\,\varepsilon\omega(p) = -\,\varepsilon\sqrt{p^2 + m^2} \\ \bar{u} &= u^{\dagger}S \end{aligned}\right\} \qquad (10.5a)$$

and $\lambda(\varepsilon,\zeta)$ is an invariant function of the arguments which can have only the values + 1 or - 1. This function must be calculated in particular cases.

We have also an orthogonality relation

$$u^{\dagger}(\underset{\sim}{p},\varepsilon,\zeta)S\gamma_0 u(\underset{\sim}{p},\,-\,\varepsilon,\zeta') = 0$$

since $S\gamma_0$ commutes with H. Hence

$$i\bar{u}(\underset{\sim}{p},\varepsilon,\zeta)\gamma^0 u(\underset{\sim}{p},\varepsilon',\zeta') = \varepsilon\lambda(\varepsilon,\zeta)\omega(p)\delta_{\varepsilon\varepsilon'}\delta_{\zeta\zeta'} \qquad (10.6)$$

which implies the completeness relation

$$\sum_{\varepsilon,\zeta} \frac{u(\underset{\sim}{p},\varepsilon,\zeta)\bar{u}(\underset{\sim}{p},\varepsilon,\zeta)}{\varepsilon\lambda(\varepsilon,\zeta)\omega(p)} = i\gamma^0 \quad . \qquad (10.7)$$

Let $u^{(s)}(\underset{\sim}{p},\varepsilon,\zeta)$ and $\bar{u}^{(s)}(\underset{\sim}{p},\varepsilon,\zeta)$ be the single spin eigenfunctions. (i.e., the last 2s - 1 components are zero.) Then if $\psi^{(s)}(x)$ is a single spin field operator satisfying the equation $(\gamma^{\mu}\nabla_{\mu} + m)\psi^{(s)}(x) = 0$, the following commutation rules are covariant.

$$[\psi^{(s)}(x),\bar{\psi}^{(s)}(y)] = C'^{-1}\,D^{(s)}(x - y) \qquad (10.8)$$

where

$$D^{(s)}(x) = -\frac{1}{(2\pi)^3}\sum_{\varepsilon,\zeta}\int \frac{u^{(s)}(\underset{\sim}{p},\varepsilon,\zeta)\bar{u}^{(s)}(\underset{\sim}{p},\varepsilon,\zeta)}{\varepsilon\lambda(\varepsilon,\zeta)\,\omega(p)} \cdot \exp(ip^{\mu}x_{\mu})\, d^3p \quad . \tag{10.9}$$

Let us define $u^{(s-1)}$ as the spin eigenfunctions orthogonal to $u^{(s)}$. These are the remaining eigenfunctions which span the space. We also define $D^{(s-1)}$ in a manner analogous to the definition of $D^{(s)}$ but use $u^{(s-1)}$ instead of $u^{(s)}$. Then

$$D^{(s)}(x) + D^{(s-1)}(x) = -i(\gamma^{\mu}\nabla_{\mu} - m)\triangle(x) \tag{10.10}$$

on using (10.7). This is expected from (6.3).

If in (10.9) the summation is taken only over a subset of the ζ's then we obtain a function $\dddot{D}^{(s)}(x)$ which when substituted in (10.8) still yields covariant commutation relations.

We can show that $D^{(s)}$ is a causal operator. Consider any 8s-dimensional column vector $f(x)$. It can always be split up in the following way:

$$f(x) = f^{(s)}(x) + f^{(s-1)}(x) \quad , \tag{10.11}$$

where $f^{(s)}(x)$ and $f^{(s-1)}(x)$ are superpositions of the $u^{(s)}$ and $u^{(s-1)}$ eigenfunctions respectively. Since the operator $D^{(s)}$ transforms the $u^{(s-1)}$ eigenfunctions to zero, we have where

[illegible]

[illegible]

[illegible]

[illegible]

on using [illegible]

[illegible]

independent column vectors [illegible]

[illegible]

[illegible]

[illegible]

$$\int D^{(s)}(x - x')f(x')dx' = \int D^{(s)}(x - x')f^{(s)}(x')dx'$$
$$= \int D(x - x')f^{(s)}(x')dx' \ . \tag{10.12}$$

Since D is a causal operator when it acts on $f^{(s)}(x)$, the causality of $D^{(s)}$ follows.

11. Example: s = 1

A set of matrices γ_λ, $K_{\mu\nu}$, and T satisfying the desired relations is given in the appendix.

Since the Hamiltonian

$$H = - i\gamma_0(\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} + m) \tag{11.1}$$

$- i\nabla$, and $- i\underset{\sim}{K} \cdot \underset{\sim}{\nabla}$ all commute, we can choose a complete orthogonal set of vectors which are simultaneously eigenvectors of H, $- i\underset{\sim}{\nabla}$ and $- i\underset{\sim}{K} \cdot \underset{\sim}{\nabla}$. Thus $\psi = ue^{ip^\mu x_\mu}$, where

$$\underset{\sim}{K} \cdot \underset{\sim}{p}u = - ip\zeta u \qquad (\zeta = \pm 1, \text{ or } 0) \quad , \tag{11.2}$$

$p = |\underset{\sim}{p}|$ and

$$H(\underset{\sim}{p})u = \varepsilon\omega u \qquad \varepsilon = \pm 1 \tag{11.3}$$

and $\omega = \sqrt{p^2 + m^2}$. To be specific let us append some indices to u so $u = u(\underset{\sim}{p},\varepsilon,\zeta)$. Also, for convenience let us put

$$(u_4(\underset{\sim}{p},\varepsilon,\zeta),u_5(\underset{\sim}{p},\varepsilon,\zeta),u_6(\underset{\sim}{p},\varepsilon,\zeta)) = \underset{\sim}{e}(\underset{\sim}{p},\varepsilon,\zeta) \ . \qquad (11.4)$$

Then equation (11.2) implies that

$$\underset{\sim}{p} \times \underset{\sim}{e}(\underset{\sim}{p},\varepsilon,\zeta) = - ip\zeta \underset{\sim}{e}(\underset{\sim}{p},\varepsilon,\zeta) \ . \qquad (11.5)$$

Since $\underset{\sim}{e}(\underset{\sim}{p},\varepsilon,\zeta)$ does not depend on ε, the ε can be dropped. In the appendix an explicit form of $\underset{\sim}{e}(\underset{\sim}{p},\varepsilon)$ is presented which satisfies the following conditions:

$$\underset{\sim}{e}^*(\underset{\sim}{p},\zeta) \cdot \underset{\sim}{e}(\underset{\sim}{p},\zeta) = 1 \ . \qquad (11.6)$$

$$\underset{\sim}{e}(\underset{\sim}{p},0) = p^{-1}\underset{\sim}{p} \ . \qquad (11.7)$$

$$\underset{\sim}{e}^*(\underset{\sim}{p},\zeta) = \underset{\sim}{e}(\underset{\sim}{p},- \zeta) \ . \qquad (11.8)$$

$$\underset{\sim}{e}(- \underset{\sim}{p}, - \zeta) = - \underset{\sim}{e}(\underset{\sim}{p},\zeta) \ . \qquad (11.9)$$

Also it follows

$$\underset{\sim}{e}^*(\underset{\sim}{p},\eta) \times \underset{\sim}{e}(\underset{\sim}{p},\eta) = i\eta \underset{\sim}{e}(\underset{\sim}{p},0) \ , \qquad \eta = \pm 1 \ . \qquad (11.10)$$

The eighth row of the wave equation says

$$\frac{\partial\psi_4}{\partial x} + \frac{\partial\psi_5}{\partial y} + \frac{\partial\psi_6}{\partial z} + \frac{\partial\psi_7}{\partial t} = 0 \ . \qquad (11.11)$$

Consequently, since $p^0 = \varepsilon\omega$,

$$u_7(\underset{\sim}{p},\varepsilon,\zeta) = \frac{\varepsilon}{\omega}\,\underset{\sim}{p}\cdot\underset{\sim}{e}(\underset{\sim}{p},\zeta) \quad . \tag{11.12}$$

If we put

$$(\psi_1,\psi_2,\psi_3) = \underset{\sim}{\psi} \quad ,$$

$$(u_1\ ,u_2,u_3) = \underset{\sim}{u} \quad ,$$

$$(\psi_4,\psi_5,\psi_6) = \underset{\sim}{A} \quad .$$

and

$$\psi_7 = V$$

we find from the first three lines of the wave equation that

$$- m\underset{\sim}{\psi} = \frac{\partial \underset{\sim}{A}}{\partial t} + \nabla V + i\nabla \times \underset{\sim}{A} \quad . \tag{11.13}$$

Consequently

$$\begin{aligned} - m\underset{\sim}{u}(\underset{\sim}{p},\varepsilon,\zeta) &= - i\varepsilon\,\omega\underset{\sim}{e}(\underset{\sim}{p},\zeta) + \frac{i\varepsilon}{\omega}\,\underset{\sim}{p}\,\underset{\sim}{p}\cdot\underset{\sim}{e}(\underset{\sim}{p},\zeta) \\ &\qquad - \underset{\sim}{p} \times \underset{\sim}{e}(\underset{\sim}{p},\zeta) \\ &= - i\varepsilon(\omega - \varepsilon\zeta p)\underset{\sim}{e}(\underset{\sim}{p},\zeta) + \frac{i\varepsilon}{\omega}\,\underset{\sim}{p}\,\underset{\sim}{p}\cdot\underset{\sim}{e}(\underset{\sim}{p},\zeta) \, . \end{aligned}$$

Hence

$$u(\underset{\sim}{p},\varepsilon,\zeta) = \begin{pmatrix} \frac{i\varepsilon}{m}(\omega - \varepsilon p\zeta)\underset{\sim}{e}(\underset{\sim}{p},\zeta) - \frac{i\varepsilon}{m\omega}\,\underset{\sim}{p}\,\underset{\sim}{p}\cdot\underset{\sim}{e}(\underset{\sim}{p},\zeta) \\ \underset{\sim}{e}(\underset{\sim}{p},\zeta) \\ \frac{\varepsilon}{\omega}\,\underset{\sim}{p}\cdot\underset{\sim}{e}(\underset{\sim}{p},\zeta) \\ 0 \end{pmatrix} \tag{11.14}$$

Instead of normalizing u, we have normalized e. The explicit forms for $\underset{\sim}{e}$ and u are in the appendix.

In addition to the bilinear covariants discussed earlier, we can also form other bilinear covariants. Let

$$b = iI\nabla_0 - 2i\,\underset{\sim}{\mathcal{M}} \cdot \underset{\sim}{\nabla} \tag{11.15}$$

let Γ be any 8-dimensional matrix which commutes with b,

$$[b,\Gamma]_- = 0 \quad , \tag{11.16}$$

and let

$$J = \psi^\dagger \beta b \Gamma \psi \quad , \qquad \beta = - i\gamma_0 \quad . \tag{11.17}$$

Then from the relation

$$[b,K]_- = - 2i\,\underset{\sim}{\mathcal{M}} \times \underset{\sim}{\nabla} \quad , \tag{11.18}$$

and

$$[b,\underset{\sim}{\mathcal{K}}]_+ = - iI\underset{\sim}{\nabla} - b\gamma_0\underset{\sim}{\gamma} + 2\,\underset{\sim}{\mathcal{M}}\nabla_0 \tag{11.19}$$

which holds only for the present case, we find that under the infinitesimal Lorentz transformation (2.3)

$$\delta J = \frac{1}{2}\omega^{\mu\nu}\psi^\dagger\beta b[K_{\mu\nu},\Gamma]_-\psi \quad . \tag{11.20}$$

Putting $\Gamma = I$ we find that $\psi^\dagger \beta b \psi$ is a scalar and putting $\Gamma = \gamma_\lambda$ we find that $\psi^\dagger \beta b \gamma_\lambda \psi$ is a 4-vector. Under time reversal $\psi^\dagger \beta b \psi \longrightarrow (\psi^\dagger \beta b \psi)^*$. Consequently $\mathrm{Re}(\psi^\dagger \beta b \psi)$ is a regular scalar. Thus we can put

$$2C' \underset{\sim}{\dot{L}} = (\beta b \psi)^\dagger (\gamma^\mu \nabla_\mu + m)\psi + \left\{(\gamma^\mu \nabla_\mu + m)\psi\right\}^\dagger \beta b \psi \tag{11.21}$$

This Lagrangian leads to the correct equation of motion if we cancel b from the Euler equations. The usual gauge invariance techniques for deriving currents leads to

$$j_\mu \sim \mathrm{Re}\ \psi^\dagger \beta b \gamma_\mu \psi \quad . \tag{11.22}$$

This Lagrangian, however, is not satisfactory for introducing the electromagnetic interaction in the usual way because it leads to a second order equation.

Incidently, in the doubled theory, the current (11.22) is proportional to the current $i\psi^\dagger S \gamma_\mu \psi$ where S is the space inversion operator.

A special case of the undoubled theory corresponds to neutral vector meson theory, while a special case of the doubled theory corresponds to the charged vector meson theory. [8,9]

To obtain the neutral vector meson theory from the undoubled equations we require $\underset{\sim}{A} = (\psi_4, \psi_5, \psi_6)$ and $V = \psi_7$ to be real.

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[8] We should like to thank Dr. M. S. Watanabe for suggesting this identification.

[9] A special case of the theory could equally well be identified with the ordinary pseudo-vector meson theory.

Then we take (letting $\underset{\sim}{\psi} = (\psi_1,\psi_2,\psi_3)$)

$$\left.\begin{aligned} &\underset{\sim}{E} = \operatorname{Re} \underset{\sim}{\psi} \\ \text{and} \qquad & \\ &\underset{\sim}{H} = - \operatorname{Im} \underset{\sim}{\psi} \quad . \end{aligned}\right\} \tag{11.24}$$

We shall now write the Dirac-like equations explicitly and separate real and imaginary parts.

The first three undoubled equations give

$$\left.\begin{aligned} - m\underset{\sim}{E} &= \frac{\partial \underset{\sim}{A}}{\partial t} + \underset{\sim}{\nabla} V \\ m\underset{\sim}{H} &= \underset{\sim}{\nabla} \times \underset{\sim}{A} \quad . \end{aligned}\right\} \tag{11.25}$$

The second three equations are

$$\left.\begin{aligned} - m\underset{\sim}{A} &= - \frac{\partial \underset{\sim}{E}}{\partial t} + \underset{\sim}{\nabla} \times \underset{\sim}{H} \\ 0 &= \frac{\partial \underset{\sim}{H}}{\partial t} + \underset{\sim}{\nabla} \times \underset{\sim}{E} \quad . \end{aligned}\right\} \tag{11.26}$$

The seventh equations yield

$$\left.\begin{aligned} - mV &= \underset{\sim}{\nabla} \cdot \underset{\sim}{E} \\ 0 &= \underset{\sim}{\nabla} \cdot \underset{\sim}{H} \quad . \end{aligned}\right\} \tag{11.27}$$

Finally the eighth equation is the Lorentz condition:

$$0 = \underset{\sim}{\nabla} \cdot \underset{\sim}{A} + \frac{\partial V}{\partial t} \quad . \tag{11.28}$$

On writing

$$\left.\begin{aligned} \underset{\sim}{E} &= (F^{01}, F^{02}, F^{03}) \\ \underset{\sim}{H} &= (F^{23}, F^{31}, F^{12}) \quad , \end{aligned}\right\} \tag{11.29}$$

equations (11.25) are equivalent to

$$mF_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{11.30}$$

where $A^0 = V$.

The first of equations (11.26) and (11.27) can be written

$$\nabla^\mu F_{\mu\nu} = mA_\nu \tag{11.31}$$

while (11.28) is

$$\nabla^\mu A_\mu = 0 \quad . \tag{11.32}$$

From (11.30), (11.31) and (11.32) one obtains

$$(\nabla^\mu \nabla_\mu - m^2) A_\nu = 0 \tag{11.33}$$

as required. One could, of course, have obtained (11.33) directly from the Dirac type equation in the usual way by noting that

all components of ψ must obey the Gordon-Klein equation.

The remaining set of equations, namely the second set of (11.26) and (11.25) can be written as the set

$$\nabla_\mu F_{\nu\kappa} + \nabla_\nu F_{\kappa\mu} + \nabla_\kappa F_{\mu\nu} = 0 \quad . \tag{11.34}$$

We shall now introduce the doubled theory in order to have a space inversion operator S as in the appendix.

In the doubled theory let $\hat{\psi}$ be written

$$\hat{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{11.35}$$

where ψ_1 and ψ_2 are eight component column vectors which satisfy

$$\left.\begin{aligned} (\gamma^\mu \nabla_\mu + m)\psi_1 &= 0 \\ (\gamma^{\mu *} \nabla_\mu + m)\psi_2 &= 0 \quad , \end{aligned}\right\} \tag{11.36}$$

and where the last component of each column vector is required to be zero. In analogy to the undoubled theory let us write

$$\psi_1 = \begin{pmatrix} \underset{\sim}{\psi}_1 \\ \underset{\sim}{A}_1 \\ V_1 \\ 0 \end{pmatrix} \quad , \qquad \psi_2 = \begin{pmatrix} \underset{\sim}{\psi}_2 \\ \underset{\sim}{A}_2 \\ V_2 \\ 0 \end{pmatrix} \quad . \tag{11.37}$$

To obtain the neutral vector meson theories we require solutions of (11.36) such that

$$\left.\begin{aligned} &\underset{\sim}{A}_1 = \underset{\sim}{A}_2 = \underset{\sim}{A} \\ &V_1 = V_2 = V \quad . \end{aligned}\right\} \tag{11.38}$$

where $\underset{\sim}{A}$ and V are real.

We can show

$$\underset{\sim}{\psi}_2 = \underset{\sim}{\psi}_1^* \quad . \tag{11.39}$$

Hence

$$\underset{\sim}{E} = \frac{\underset{\sim}{\psi}_2 + \underset{\sim}{\psi}_1}{2} \quad , \qquad \underset{\sim}{H} = \frac{\underset{\sim}{\psi}_2 - \underset{\sim}{\psi}_1}{2i} \; . \tag{11.40}$$

In the case of the doubled neutral vector meson theory, we can show that the current $j_\mu \sim i\psi^* S\gamma_\mu\psi \equiv 0$.

To obtain the charged vector meson field we must use the doubled theory as in the neutral theory. But now we require $\underset{\sim}{A}$ and V of (11.38) to be complex. The vectors $\underset{\sim}{E}$ and $\underset{\sim}{H}$ are *defined* by (11.40) and satisfy the same equations in terms of $\underset{\sim}{A}$ and V (11.25) - (11.34) in the neutral theory. In contrast to the neutral theory $j_\mu \neq 0$, in general.

We can now quantize the theory.

For the charged meson field we have

$$\begin{aligned}\psi_1(\underset{\sim}{x},t) = C'' \int d^3p \Big\{ \Big[\sqrt{\frac{\omega}{m}}\, u(\underset{\sim}{p},+,0)\, a(\underset{\sim}{p},+,0) + \sum_{\zeta=\pm 1} \sqrt{\frac{m}{\omega}}\, u(p,+,\zeta) \\ \cdot\, a(\underset{\sim}{p},+,\zeta) \Big] e^{ip_\mu x^\mu} + \Big[\sqrt{\frac{\omega}{m}}\, u(\underset{\sim}{p},-,0)\, a^\dagger(\underset{\sim}{p},-,0) + \sum_{\zeta=\pm 1} \sqrt{\frac{m}{\omega}} \\ \cdot\, u(\underset{\sim}{p},-,-\zeta)\, a^\dagger(\underset{\sim}{p},-,\zeta) \Big] e^{-ip_\mu x^\mu} \Big\}\end{aligned} \tag{11.40}$$

while for the neutral meson field

$$\begin{aligned}\psi(x,t) = C''' \int d^3p \Big\{ \Big[\sqrt{\frac{\omega}{m}}\, u(\underset{\sim}{p},+,0)\, a(\underset{\sim}{p},0) + \sum_{\zeta=\pm 1} \sqrt{\frac{m}{\omega}}\, u(\underset{\sim}{p},+,\zeta) \\ \cdot\, a(\underset{\sim}{p},\zeta) \Big] e^{ip_\mu x^\mu} + \Big[\sqrt{\frac{\omega}{m}}\, u(\underset{\sim}{p},-,0)\, a^\dagger(\underset{\sim}{p},0) + \sum_{\zeta=\pm 1} \sqrt{\frac{m}{\omega}} \\ \cdot\, u(\underset{\sim}{p},-,-\zeta)\, a^\dagger(\underset{\sim}{p},\zeta) \Big] e^{-ip_\mu x^\mu} \Big\}\end{aligned} \tag{11.41}$$

where

$$[a(\underset{\sim}{p},\varepsilon,\zeta), a(\underset{\sim}{p}',\varepsilon',\zeta')]_- = 0 \tag{11.42}$$

$$[a(\underset{\sim}{p},\varepsilon,\zeta), a^\dagger(\underset{\sim}{p}',\varepsilon',\zeta')]_- = \delta_{\varepsilon\varepsilon'}\delta_{\zeta\zeta'}\delta(\underset{\sim}{p} - \underset{\sim}{p}') \tag{11.43}$$

$$[a(\underset{\sim}{p},\zeta), a(\underset{\sim}{p}',\zeta')]_- = 0$$

$$[a(\underset{\sim}{p},\zeta), a^\dagger(\underset{\sim}{p}',\zeta')]_- = \delta_{\zeta\zeta'}\delta(\underset{\sim}{p} - \underset{\sim}{p}') \quad . \tag{11.44}$$

Appendix I

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$K_{23} = \begin{pmatrix} 0 & 0 & 0 & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & -1 & 0 & & & & & \\ & & & 0 & 0 & 0 & & \\ & & & 0 & 0 & -1 & & \\ & & & 0 & 1 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix} \qquad K_{31} = \begin{pmatrix} 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & & & \\ -1 & 0 & 0 & & & & & \\ & & & 0 & 0 & 1 & & \\ & & & 0 & 0 & 0 & & \\ & & & -1 & 0 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$K_{12} = \begin{pmatrix} 0 & -1 & 0 & & & & & \\ 1 & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ & & & 0 & -1 & 0 & & \\ & & & 1 & 0 & 0 & & \\ & & & 0 & 0 & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$K_{01} = \begin{pmatrix} 0 & 0 & 0 & & & & & \\ 0 & 0 & -i & & & & & \\ 0 & i & 0 & & & & & \\ & & & 0 & 0 & 0 & -1 & \\ & & & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 0 & \\ & & & -1 & 0 & 0 & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$K_{02} = \begin{pmatrix} 0 & 0 & i & & & & & \\ 0 & 0 & 0 & & & & & \\ -i & 0 & 0 & & & & & \\ & & & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & -1 & \\ & & & 0 & 0 & 0 & 0 & \\ & & & 0 & -1 & 0 & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$K_{03} = \begin{pmatrix} 0 & -i & 0 & & & & & \\ i & 0 & 0 & & & & & \\ 0 & 0 & 0 & & & & & \\ & & & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & 0 & \\ & & & 0 & 0 & 0 & -1 & \\ & & & 0 & 0 & -1 & 0 & \\ & & & & & & & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix}$$

$$\hat{S} = \begin{pmatrix} 0_8 & \begin{matrix} -1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{matrix} \\ \begin{matrix} -1 & & & & & & & \\ & -1 & & & & & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{matrix} & 0_8 \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} O_8 & I_8 \\ I_8 & O_8 \end{pmatrix}$$

$$\hat{S}_1 = \begin{pmatrix} O_8 & \begin{matrix} -1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} \\ \begin{matrix} -1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} & O_8 \end{pmatrix}$$

$$\hat{S}_2 = \begin{pmatrix} O_8 & \begin{matrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} \\ \begin{matrix} 1 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& 1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} & O_8 \end{pmatrix} \qquad \hat{S}_3 = \begin{pmatrix} O_8 & \begin{matrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} \\ \begin{matrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && -1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& -1 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{matrix} & O_8 \end{pmatrix}$$

$$\left.\begin{aligned} \underset{\sim}{e}(\underset{\sim}{p},0) &= \frac{\underset{\sim}{p}}{p} \\ e_1(\underset{\sim}{p},\zeta) &= \frac{pp_3 - i\zeta p_1p_2}{p\sqrt{2(p_1^2+p_3^2)}} \\ e_2(\underset{\sim}{p},\zeta) &= \frac{i\zeta}{p}\sqrt{\frac{p_1^2+p_3^2}{2}} \\ e_3(\underset{\sim}{p},\zeta) &= \frac{-(pp_1 + i\zeta p_3p_2)}{p\sqrt{2(p_1^2+p_3^2)}} \\ p &= |\underset{\sim}{p}| \quad . \end{aligned}\right\} \qquad \zeta = \pm 1$$

$$u(\underset{\sim}{p},\varepsilon,0) = \begin{pmatrix} \frac{i\varepsilon\ mp_1}{\omega p} \\ \frac{i\varepsilon\ mp_2}{\omega p} \\ \frac{i\varepsilon\ mp_3}{\omega p} \\ \frac{p_1}{p} \\ \frac{p_2}{p} \\ \frac{p_3}{p} \\ \frac{\varepsilon p}{\omega} \\ 0 \end{pmatrix}$$

$$u(\underset{\sim}{p},\varepsilon,\zeta) = \begin{pmatrix} \frac{i\varepsilon}{m}(\omega - \varepsilon\zeta p)e_1(\underset{\sim}{p},\zeta) \\ \frac{i\varepsilon}{m}(\omega - \varepsilon\zeta p)e_2(\underset{\sim}{p},\zeta) \\ \frac{i\varepsilon}{m}(\omega - \varepsilon\zeta p)e_3(\underset{\sim}{p},\zeta) \\ e_1(\underset{\sim}{p},\zeta) \\ e_2(\underset{\sim}{p},\zeta) \\ e_3(\underset{\sim}{p},\zeta) \\ 0 \\ 0 \end{pmatrix} , \quad \zeta = \pm 1 .$$

Appendix II

List of Improper Transformations

$T = e^{i\phi} V^{-1} D \gamma'_2 \gamma'_3 V^{-1t}$

$S = \pm\, i V^{-1} \rho' \gamma'_0 V$ for undoubled theory,

$S = \begin{pmatrix} 0 & e^{i\phi} A \\ e^{-i\phi} A^{\dagger} & 0 \end{pmatrix}$ for doubled theory,

where

$A = i V^{-1} D \gamma'_0 V^{*}$

$C = -\, i e^{i\phi} V^{-1} \rho' D \cdot V^{*}$ for undoubled theory.

$C = e^{i\frac{\pi}{2}(2m+2n+1)} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ for doubled theory.

The operators D, ρ', γ'_μ, V, are defined in Section 3.

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